JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 167, 66–73 (1992)

Upper Limits to the Complex Growth Rates in Thermohaline Instability

MIHIR B. BANERJEE, J. R. GUPTA, AND JYOTI PRAKASH

Department of Mathematics, Himachal Pradesh University, Shimla, India

Submitted by E. Stanley Lee

Received September 10, 1990

The paper presents a significant improvement on Banerjee *et al.*'s results [*Proc. Roy. Soc. London Ser. A* **378** (1981), 301–304] which prescribe upper limits to the complex growth rates of arbitrary oscillatory motions of growing amplitude in thermohaline instability. The modified results derived here are of wider generality and applicability than the simple context in which they are presented and lead in the context of Veronis' configuration to an alternative nondimensional system of governing equations and boundary conditions of thermohaline instability in terms of a new nondimensional number B, on which the nonexistence of oscillatory motions of growing amplitude crucially depends.

1. INTRODUCTION

The hydrodynamic instability that manifests under appropriate conditions in a static horizontal initially homogeneous viscous and Boussinesq liquid layer of infinite horizontal extension and finite vertical depth which is kept under the simultaneous action of a uniform vertical temperature gradient and a gravitationally opposite uniform vertical concentration gradient in the force field of gravity is known as thermohaline instability. Two fundamental configurations have been studied in this context, the first one by Stern [2] wherein the temperature gradient is stabilizing while the concentration gradient is destabilizing and the second one by Veronis [3] wherein the temperature gradient is destabilizing while the concentration gradient is stabilizing. The main results derived by Stern and Veronis for the respective problems are that instability might occur in the configurations through a stationary pattern of motions or oscillatory motions provided the destabilizing concentration gradient or temperature gradient is sufficiently large but compatible with the condition that the total density field is gravitationally stable. The problem of obtaining bounds for the complex growth rate of an arbitrary oscillatory motion of growing amplitude in the above two configurations is important especially when

0022-247X/92 $5.00

both the boundaries are not dynamically free so that exact solutions in closed forms are not obtainable as was possible for the cases treated by Stern and Veronis. Banerjee *et al.* [1] formulated a novel way of combining the governing equations and boundary conditions for each of the two problems so that a semicircle theorem is derivable in each case and which in turn yields the desired bounds. However, the independence of the semicircle theorem, for Veronis' configuration in particular, wherein oscillatory motions of growing amplitude are the preferred ones at the marginal state, of the Lewis number τ which expresses the ratio of mass diffusivity to heat diffusivity raises doubts about the accuracy of the concerned bound and it appears that a more rigid limitation than Banerjee *et al.*'s can be obtained on the concerned growth rate by the inclusion of τ in the mathematical derivation of the semicircle theorem. The present paper is precisely based on this contention and provides a significant improvement in the results of Banerjee *et al.* [1] which prescribe upper limits to the complex growth, rates of arbitrary oscillatory motions of growing amplitude in thermohaline instability. The modified results derived here are of wider generality and applicability than the simple context in which they are presented and lead in the context of Veronis' configuration to an alternative nondimensional system of governing equations and boundary conditions of thermohaline instability in terms of a new nondimensional number B, on which nonexistence of oscillatory motions of growing amplitude crucially depends.

2. Mathematical Formulation and Analysis

The relevant governing equations and boundary conditions of thermohaline instability are given by [1]

$$(D^2-a^2)(D^2-a^2-p/\sigma)\,w=Ra^2\theta-R_s a^2\phi \tag{1}$$

$$(D^2-a^2-p)\,\theta=-w \tag{2}$$

$$(D^2-a^2-p/\tau)\,\phi=-\frac{w}{\tau} \tag{3}$$

with

$$w=0=\theta=\phi=D^2w \quad \text{at} \quad z=0 \text{ and } z=1 \tag{4}$$

or

$$w=0=\theta=\phi=Dw \quad \text{at} \quad z=0 \text{ and } z=1 \tag{5}$$

or

$$w=0=\theta=\phi=Dw \text{ at } z=0 \quad \text{and} \quad w=0=\theta=\phi=D^2w \text{ at } z=1 \tag{6}$$

or

$$w = 0 = \theta = \phi = D^2 w \text{ at } z = 0 \qquad \text{and} \qquad w = 0 = \theta = \phi = Dw \text{ at } z = 1 \tag{7}$$

where z is a real independent variable such that $0 \leqslant z \leqslant 1$, $D = d/dz$ is differentiation with respect to z, $a^2 > 0$ is a constant, $\sigma > 0$ is a constant, $\tau > 0$ is a constant, R and R_s are positive constants for Veronis' configuration and negative constants for Stern's configuration, $p = p_r + ip_i$ is a complex constant such that p_r and p_i are real constants and as a consequence the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$, and $\phi(z) = \phi_r(z) + i\phi_i(z)$ are complex valued functions of the real variable z such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_r(z)$, and $\phi_i(z)$ are real valued functions of the real variable z. The meanings of the symbols from the physical point of view are as follows: z is the vertical coordinate, d/dz is differentiation along the vertical direction, a^2 is the square of the wave number, σ is the Prandtl number, τ is the Lewis number, R is the Rayleigh number, R_s is the concentration Rayleigh number, p is the complex growth rate, w is the vertical velocity, θ is the temperature, and ϕ is the concentration. It may further be noted that Eqs. (1)–(7) describe an eigenvalue problem for p and govern thermohaline instability for any combination of dynamically free and rigid boundaries.

We prove the following theorems:

THEOREM 1. *If $R > 0$, $R_s > 0$, $p_r \geqslant 0$, and $p_i \neq 0$, then a necessary condition for the existence of a nontrivial solution (w, θ, ϕ, p) of Eqs.* (1)–(3) *with boundary conditions* (4), (5), (6), *or* (7) *is that*

$$|p|^2 < R_s \sigma - 2\tau^2 \pi^4. \tag{8}$$

Proof. Multiplying Eq. (1) by w^* (the complex conjugate of w) throughout and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) w \, dz = Ra^2 \int_0^1 w^*\theta \, dz - R_s a^2 \int_0^1 w^*\phi \, dz. \tag{9}$$

Making use of Eqs. (2) and (3), we can write

$$Ra^2 \int_0^1 w^*\theta \, dz = -Ra^2 \int_0^1 \theta(D^2 - a^2 - p^*)\,\theta^* \, dz \tag{10}$$

$$-R_s a^2 \int_0^1 w^*\phi \, dz = R_s a^2 \tau \int_0^1 \phi\left(D^2 - a^2 - \frac{p^*}{\tau}\right)\phi^* \, dz. \tag{11}$$

Combining Eqs. (9)–(11), we obtain

$$\int_0^1 w^*(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right) w\,dz$$

$$= -Ra^2\int_0^1 \theta(D^2-a^2-p^*)\,\theta^*\,dz$$

$$+ R_s a^2\tau\int_0^1 \phi\left(D^2-a^2-\frac{p^*}{\tau}\right)\phi^*\,dz. \tag{12}$$

Integrating the various terms of Eq. (12) by parts for an appropriate number of times and making use of any of the boundary conditions (4)–(7), it follows that

$$\int_0^1 (|D^2w|^2+2a^2|Dw|^2+a^4|w|^2)\,dz+\frac{p}{\sigma}\int_0^1 (|Dw|^2+a^2|w|^2)\,dz$$

$$= Ra^2\int_0^1 (|D\theta|^2+a^2|\theta|^2+p^*|\theta|^2)\,dz$$

$$-R_s a^2\tau\int_0^1 \left(|D\phi|^2+a^2|\phi|^2+\frac{p^*}{\tau}|\phi|^2\right)dz. \tag{13}$$

Equating the imaginary parts of both sides of Eq. (13) and cancelling p_i ($\neq 0$) throughout from the resulting equation, we get

$$\frac{1}{\sigma}\int_0^1 |Dw|^2\,dz+\frac{a^2}{\sigma}\int_0^1 |w|^2\,dz = -Ra^2\int_0^1 |\theta|^2\,dz+R_s a^2\int_0^1 |\phi|^2\,dz. \tag{14}$$

Now, from Eq. (2) we derive that

$$\int_0^1 (D^2-a^2-p)\,\theta\cdot(D^2-a^2-p^*)\,\theta^*\,dz=\int_0^1 |w|^2\,dz. \tag{15}$$

Integrating the various terms on the left hand side of Eq. (15) by parts an appropriate number of times and making use of the boundary conditions on θ, we have

$$\int_0^1 (|D^2\theta|^2+2a^2|D\theta|^2+a^4|\theta|^2)\,dz+2p_r\int_0^1 (|D\theta|^2+a^2|\theta|^2)\,dz$$

$$+|p|^2\int_0^1 |\theta|^2\,dz=\int_0^1 |w|^2\,dz \tag{16}$$

which can be written as

$$I_1^2 + |p|^2 \int_0^1 |\theta|^2 \, dz = \int_0^1 |w|^2 \, dz, \tag{17}$$

where

$$I_1^2 = \int_0^1 (|D^2\theta|^2 + 2a^2 |D\theta|^2 + a^4 |\theta|^2) \, dz$$
$$+ 2p_r \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) \, dz \tag{18}$$

is positive definite.

Similarly from Eq. (3) we derive that

$$\int_0^1 \left(D^2 - a^2 - \frac{p}{\tau}\right) \phi \cdot \left(D^2 - a^2 - \frac{p^*}{\tau}\right) \phi^* \, dz = \frac{1}{\tau^2} \int_0^1 |w|^2 \, dz \tag{19}$$

and integrating the various terms on the left hand side of Eq. (19) by parts an appropriate number of times and making use of the boundary conditions on ϕ, we have

$$\int_0^1 (|D^2\phi|^2 + 2a^2 |D\phi|^2 + a^4 |\phi|^2) \, dz + \frac{2p_r}{\tau} \int_0^1 (|D\phi|^2 + a^2 |\phi|^2) \, dz$$
$$+ \frac{|p|^2}{\tau^2} \int_0^1 |\phi|^2 \, dz = \frac{1}{\tau^2} \int_0^1 |w|^2 \, dz \tag{20}$$

which can be written as

$$I_2^2 + \frac{|p|^2}{\tau^2} \int_0^1 |\phi|^2 \, dz = \frac{1}{\tau^2} \int_0^1 |w|^2 \, dz, \tag{21}$$

where

$$I_2^2 = \int_0^1 (|D^2\phi|^2 + 2a^2 |D\phi|^2 + a^4 |\phi|^2) \, dz$$
$$+ \frac{2p_r}{\tau} \int_0^1 (|D\phi|^2 + a^2 |\phi|^2) \, dz \tag{22}$$

is positive definite.

Combining Eqs. (14), (17), and (21), we derive that

$$\frac{1}{\sigma}\int_0^1 |Dw|^2\,dz+\frac{a^2}{\sigma}\int_0^1 |w|^2\,dz$$
$$=\frac{-Ra^2}{|p|^2}\int_0^1 |w|^2\,dz+\frac{R_s a^2}{|p|^2}\int_0^1 |w|^2\,dz$$
$$+\frac{Ra^2}{|p|^2}I_1^2-\frac{R_s a^2\tau^2}{|p|^2}I_2^2. \tag{23}$$

Further, it follows from Eqs. (14), (17), and (22) that

$$a^2\int_0^1 |\phi|^2\,dz>\frac{1}{R_s\sigma}\int_0^1 |Dw|^2\,dz \tag{24}$$

$$I_1^2<\int_0^1 |w|^2\,dz \tag{25}$$

$$I_2^2>2a^2\int_0^1 |D\phi|^2\,dz. \tag{26}$$

Also, since $w(0)=0=w(1)$ and $\phi(0)=0=\phi(1)$, therefore by the Rayleigh–Ritz inequality (Schultz [4]) we have

$$\int_0^1 |Dw|^2\,dz\geqslant\pi^2\int_0^1 |w|^2\,dz \tag{27}$$

$$\int_0^1 |D\phi|^2\,dz\geqslant\pi^2\int_0^1 |\phi|^2\,dz. \tag{28}$$

Now, making use of inequalities (24)–(28), we can write Eq. (23) as

$$\frac{1}{\sigma}\int_0^1 |Dw|^2\,dz+\frac{a^2}{\sigma}\int_0^1\left[1+\frac{(2\tau^2\pi^4-R_s\sigma)}{|p|^2}\right]|w|^2\,dz<0 \tag{29}$$

and it follows from inequality (29) that

$$|p|^2<R_s\sigma-2\tau^2\pi^4, \tag{30}$$

which establishes the required result.

Theorem 1 may be stated in an equivalent form as: the complex growth rate of an arbitrary oscillatory motion of growing amplitude, in thermohaline instability of the Veronis type, lies inside a semicircle in the upper half of the $p_r p_i$-plane whose centre is at the origin and whose radius is

$\sqrt{R_s\sigma - 2\tau^2\pi^4}$. The incorporation of the Lewis number τ in the mathematical analysis has resulted in shortening the radius of the semicircle of the corresponding circle-theorem established by Banerjee *et al.* [1] and thus the present work is definitely an improvement over their work. Further, it follows from inequality (30) that a sufficient condition for the validity of the "principle of exchange of stabilities" in thermohaline instability of the Veronis type is that

$$\frac{R_s\sigma}{2\tau^2\pi^4} \leqslant 1, \tag{31}$$

a result obtained earlier by Gupta *et al.* [5].

It is therefore clear that the existence of oscillatory motions of growing amplitude in this problem depends crucially upon the magnitude of the nondimensional number $B = R_s\sigma/2\tau^2\pi^4$ in the sense that so long as $0 < B \leqslant 1$ no such motions are possible. We then write down the governing equations and boundary conditions presented by Eqs. (1)–(7), in terms of the new nondimensional number B, as

$$(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) W = Ra^2\Theta - Ba^2\phi \tag{32}$$

$$(D^2 - a^2 - p)\,\Theta = -W \tag{33}$$

$$\left(D^2 - a^2 - \frac{p}{\tau}\right)\phi = -\frac{2\pi^4\tau}{\sigma} W, \tag{34}$$

where

$$\left.\begin{aligned} B &= \frac{R_s\sigma}{2\tau^2\pi^4} \\ W &= \frac{\sigma}{2\tau^2\pi^4} w \\ \Theta &= \frac{\sigma}{2\tau^2\pi^4} \theta \end{aligned}\right\} \tag{35}$$

with the same boundary conditions as given by Eqs. (4)–(7) with w and θ replaced respectively by W and Θ. Equations (32)–(35) are more appropriate for the investigation of oscillatory motions in thermohaline instability of the Veronis type.

For the thermohaline instability of the Stern type we prove the following theorem:

THEOREM 2. *If* $R<0$, $R_s<0$, $p_r \geqslant 0$, *and* $p_i \neq 0$, *then a necessary condition for the existence of a nontrivial solution* (w, θ, ϕ, p) *of Eqs.* (1)–(3) *with boundary conditions* (4), (5), (6), *or* (7) *is that*

$$|p|^2 < |R|\,\sigma - 2\pi^4. \tag{36}$$

Proof. If we proceed in a manner similar to the one that we adopted in proving Theorem 1, the desired result follows.

An equivalent form of the statement of Theorem 2 would be that the complex growth rate of an arbitrary oscillatory motion of growing amplitude, in a thermohaline instability of Stern type, lies inside a semicircle in the upper half of the $p_r p_i$-plane whose centre is at the origin and whose radius is $\sqrt{|R|\,\sigma - 2\pi^4}$. Theorem 2 is surely an improvement over the corresponding theorem established by Banerjee *et al.* [1]. Furter, it follows from inequality (36) that a sufficient condition for the validity of the "principle of exchange of stabilities" is that $|R|\,\sigma/2\pi^4 \leqslant 1$, a result obtained by Gupta *et al.* [5].

REFERENCES

1. M. B. BANERJEE, D. C. KATOCH, G. S. DUBE, AND K. BANERJEE, Bounds for growth rate of a perturbation in thermohaline convection, *Proc. Roy. Soc. London Ser. A* **378** (1981), 301.
2. M. E. STERN, The salt fountain and thermohaline convection, *Tellus* **12** (1960), 172.
3. G. VERONIS, On finite amplitude instability in thermohaline convection, *J. Marine Res.* **23** (1965), 1.
4. M. H. SCHULTZ, "Spline Analysis," Prentice–Hall, Englewood Cliffs, NJ, 1973.
5. J. R. GUPTA, S. K. SOOD, AND U. D. BHARDWAJ, On the characterization of nonoscillatory motions in rotatory hydromagnetic thermohaline convection, *Indian J. Pure Appl. Math.* **17** (1986), 100.